Orbiter Technical Notes: Distributed Vessel Mass

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1 Introduction

A point mass m placed at position $\vec{r}_0$ in a gravitational field $\vec{g}(\vec{r})$ experiences a force $\vec{F}_G = m\vec{g}(\vec{r}_0)$. For an extended object with a density distribution $\rho(\vec{r})$ the resulting force can be obtained by integrating over its volume $V \subset \mathbb{R}^3$:

$$\vec{F}_G = \int_V \vec{g}(\vec{r})\rho(\vec{r})d\vec{r}$$

For numerical calculations it is sometimes useful to discretise the object into a rigid system of point masses m_i whose relative positions are defined by their barycentric coordinates $\vec{s}_i$. Then,

$$\vec{F}_G = \sum_i m_i \vec{g}(\vec{s}_i + \vec{r}_{\mathrm{CG}})$$

where $\vec{r}_{\mathrm{CG}}$ is the position of the barycentre. For the calculation of the linear force $\vec{F}_G$ Orbiter makes the assumption $\vec{g}(\vec{s}_i + \vec{r}_{\mathrm{CG}}) = \vec{g}(\vec{r}_{\mathrm{CG}})$, i.e. the gravitational field is homogeneous over the volume of the object. This approximation is justified when calculating the gravitational force on a spacecraft which is small compared to its orbital radius vector, $|\vec{s}_i| \ll |\vec{r}_{\mathrm{CG}}|$. With this assumption, we arrive back at the expression for a point mass:

$$\vec{F}_G = \vec{g}(\vec{r}_{\mathrm{CG}}) \sum_i m_i = m\vec{g}(\vec{r}_{\mathrm{CG}})$$

However, an inhomogeneous potential will also induce an angular moment $\vec{M}_G$ in an extended object, and this can generally not be neglected. In the continuous case, $\vec{M}_G$ is given by

$$\vec{M}_G = \int_V \vec{g}(\vec{r})\rho(\vec{r}) \times (\vec{r} - \vec{r}_{\mathrm{CG}})d\vec{r}, \tag{1}$$

and after discretisation this becomes

$$\vec{M}_G = \sum_i m_i \vec{g}(\vec{s}_i + \vec{r}_{\mathrm{CG}}) \times \vec{s}_i \tag{2}$$

2 Symmetric gravitational potential

If we assume that

$$\vec{g}(\vec{r}) = -GMr^{-3}\vec{r}, \tag{3}$$

i.e. the central body is a sphere of mass M with homogeneous density distribution, and further that $|\vec{r}| = |\vec{r}_{\mathrm{CG}} + \vec{s}| \gg |\vec{s}|$ over the volume V of the spacecraft, then we can approximate [1]:

$$r^{-3} = (\vec{r}\cdot\vec{r})^{-3/2} = \left\{ r_{\mathrm{CG}}^2 \left[1 + \frac{2\vec{r}_{\mathrm{CG}}\cdot\vec{s}}{r_{\mathrm{CG}}^2} + \frac{\vec{s}^2}{r_{\mathrm{CG}}^2} \right] \right\}^{-3/2} \approx r_{\mathrm{CG}}^{-3} \left[1 - \frac{3\vec{r}_{\mathrm{CG}}\cdot\vec{s}}{r_{\mathrm{CG}}^2} \right] \tag{4}$$

Substituting Eqns. 3 and 4 into Eq. 1 leads to

$$\vec{M}_G = \frac{3GM}{r_{\mathrm{CG}}^3} \int_V (\hat{r}_{\mathrm{CG}} \times \vec{s})(\hat{r}_{\mathrm{CG}} \cdot \vec{s})\rho(\vec{r})d\vec{r} \tag{5}$$

If the vectors $\vec{s}$ and $\hat{r}_{\mathrm{CG}}$ are expressed in the vessel reference system, then $\vec{M}_G$ can be written as

$$\vec{M}_G = \frac{3GM}{r_{\mathrm{CG}}^3} [(\mathsf{L}\hat{r}_{\mathrm{CG}}) \times \hat{r}_{\mathrm{CG}}] \tag{6}$$

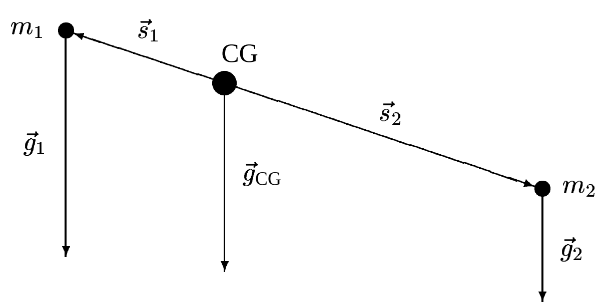
where L is the vessel's inertia tensor expressed in the same frame. (Note that Orbiter currently assumes that the vessel frame of reference is orientated so that L is diagonal.)

3 Discrete point mass systems

Orbiter implements gravity gradient torque as discussed in Section 2, assuming that the vessel's inertia tensor is known. In this section we give an alternative method that describes the vessel as a rigid system of point masses. This method is not currently implemented in Orbiter.

3.1 2-point systems

In the simplest case, a vessel can be expressed as a rigid system of two point masses. This allows to simulate an angular moment as a result of a gradient in the gravitational potential $\vec{g}(\vec{r})$.



We want to calculate the angular moment induced by the difference of the gravitational fields at $\vec{s}_1$ and $\vec{s}_2$. Let the ratio of masses be denoted by $m_1/m_2 = \gamma$. Then $\vec{s}_2 = -\gamma\vec{s}_1$, and the forces acting on the two mass points are

$$\vec{F}_1 = m_1\vec{g}_1, \qquad \vec{F}_2 = m_2\vec{g}_2 = \frac{m_1}{\gamma}\vec{g}_2.$$

The angular moment is obtained by adding both components,

$$\begin{aligned} \vec{M}_G &= \vec{F}_1 \times \vec{s}_1 + \vec{F}_2 \times \vec{s}_2 \\ &= m_1\vec{g}_1 \times \vec{s}_1 - \frac{m_1}{\gamma}\vec{g}_2 \times \gamma\vec{s}_1 \\ &= m_1 \left(\vec{g}_1 - \vec{g}_2\right) \times \vec{s}_1 \end{aligned} \tag{7}$$

which is now expressed as a function of the local field gradient $\vec{g}_1 - \vec{g}_2$.

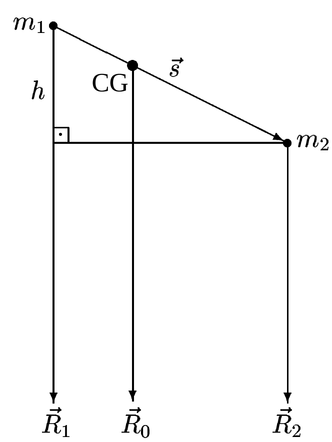
3.2 Numerical implementation

The field difference is usually small compared to the magnitude of the field acting on the vessel:

$$|\vec{g}_1 - \vec{g}_2| \ll |\vec{g}_{1,2}|$$

resulting in a significant loss of precision when the field difference in Eq. 7 is calculated directly.

To avoid this problem, we can simplify Eq. 7 if the field $\vec{g}(\vec{r})$ can be condsidered to be generated by a single point mass M at position $\vec{R}_0$ relative to CG. Let $\vec{R}_1 = \vec{R}_0 - \vec{s}_1$ and $\vec{R}_2 = \vec{R}_0 - \vec{s}_2$ be the position of M relative to m_1 and m_2, and let $\vec{s} = \vec{s}_2 - \vec{s}_1$.



Since $|\vec{R}_0| \gg |\vec{s}|$, we can find the difference $h = |\vec{R}_1| - |\vec{R}_2|$ of the distances between the point masses m_1 and m_2 from M by

$$h = |\vec{s}| \cos \sphericalangle(\vec{s}, \vec{R}_0) = \frac{\vec{s}\vec{R}_0}{|\vec{R}_0|}$$

We can now write the field difference $\vec{g}_1 - \vec{g}_2$ as

$$\vec{g}_1 - \vec{g}_2 = GM \left(\frac{\vec{R}_1}{|\vec{R}_1|^3} - \frac{\vec{R}_2}{|\vec{R}_2|^3} \right)$$

and by substituting $\vec{R}_2 = \vec{R}_1 - \vec{s}$ and $|\vec{R}_2| = |\vec{R}_1| - h$, and omitting higher-order terms,

$$\begin{aligned}
\frac{\vec{g}_1 - \vec{g}_2}{GM} &= \frac{\vec{R}_1}{|\vec{R}_1|^3} - \frac{\vec{R}_1 - \vec{s}}{(|\vec{R}_1| - h)^3} \\
&= \frac{h(-3|\vec{R}_1|^2 + 3|\vec{R}_1|h - h^2)\vec{R}_1 + |\vec{R}_1|^3\vec{s}}{|\vec{R}_1|^3(|\vec{R}_1| - h)^3} \\
&= \frac{3h(h - |\vec{R}_1|)\vec{R}_1 + |\vec{R}_1|^2\vec{s}}{|\vec{R}_1|^4(|\vec{R}_1| - 3h)} + O(2)
\end{aligned}$$

Substituting into Eq. 7 and utilising $\vec{s} \parallel \vec{s}_1$ leads to

$$\vec{M}_G \approx 3GMm_1 \frac{h(h - |\vec{R}_1|)}{|\vec{R}_1|^4(|\vec{R}_1| - 3h)} \vec{R}_1 \times \vec{s}_1$$

3.3 Multi-point systems

For objects composed of more than two mass points, the formulation must be somewhat extended. We split the gravitational potential acting on mass point m_i into a barycentric and a perturbation component: $\vec{g}(\vec{r}_{\text{CG}} + \vec{s}_i) = \vec{g}_{\text{CG}} + \vec{\gamma}_i$. Then Eq. 2 becomes

$$\begin{aligned}
\vec{M}_G &= \sum_i (\vec{g}_{\text{CG}} + \vec{\gamma}_i) \times m_i \vec{s}_i \\
&= \vec{g}_{\text{CG}} \times \sum_i m_i \vec{s}_i + \sum_i \vec{\gamma}_i \times m_i \vec{s}_i \\
&= \sum_i \vec{\gamma}_i \times m_i \vec{s}_i
\end{aligned} \tag{8}$$

since the definition of the barycentre demands $\sum_i m_i \vec{s}_i = 0$.

The perturbation components are calculated in the same way as in the 2-point problem, assuming that the gravitational potential is generated by a single point mass M at position $\vec{R}_0$ in barycentric coordinates of the spacecraft, with $|\vec{R}_0| \gg |\vec{s}_i| \ \forall i$. Given $\vec{R}_i = \vec{R}_0 - \vec{s}_i$, the difference h_i between the distances of m_i and the CG from M is given by

$$h_i = |\vec{s}_i| \cos \sphericalangle(-\vec{s}_i, \vec{R}_0) = -\frac{\vec{s}_i \vec{R}_0}{|\vec{R}_0|}$$

Then as before,

$$\begin{aligned}
\frac{\vec{\gamma}_i}{GM} &= \frac{\vec{R}_0 - \vec{s}_i}{(|\vec{R}_0| + h_i)^3} - \frac{\vec{R}_0}{|\vec{R}_0|^3} \\
&= -\frac{h_i(3|\vec{R}_0|^2 + 3|\vec{R}_0|h_i + h_i^2)\vec{R}_0 + |\vec{R}_0^3|\vec{s}_i}{(|\vec{R}_0| + h_i)^3|\vec{R}_0|^3} \\
&= -\frac{3h_i(|\vec{R}_0| + h_i)\vec{R}_0 + |\vec{R}_0|^2\vec{s}_i}{|\vec{R}_0|^4(|\vec{R}_0| + 3h_i)} + O(2)
\end{aligned}$$

and inserting into Eq. 8 leads to

$$\vec{M}_G = -\frac{3GM}{|\vec{R}_0|^4}\vec{R}_0 \times \sum_i \frac{m_i h_i(|\vec{R}_0| + h_i)}{|\vec{R}_0| + 3h_i}\vec{s}_i$$

4 Damping

The model defined above has equilibrium states ($\vec{M}_G = 0$) for the attitudes $\vec{s} \parallel \vec{R}_0$ (vessel axis aligned with radius vector) and $\vec{s} \perp \vec{R}_0$ (vessel axis perpendicular to radius vector). Only the first of these is stable because an attitude perturbation will generate a torque in the opposite direction, leading to an undamped oscillation around the equilibrium attitude.

Orbiter allows to introduce a damping term

$$\vec{M}_D = -\alpha\vec{\omega}_G$$

where $\vec{\omega}_G$ is the angular velocity induced by torque $\vec{M}_G$, and α is a user-defined damping coefficient. The physical source for the damping term may be the deformation of the vessel by tidal forces, or redistribution of liquid propellants.

References

[1] J. R. Wertz, editor. *Spacecraft Attitude Determination and Control*. Kluwer Academic Publishers, Dordrecht, 1978.